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ON THE SHOCK WAVE IN A FLOW PAST A CONVEX CORNER

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A local problem of a subsonic gas flow past a finite convex corner is studied. Complete gasdynamic equations are used to show that a shock-free flow is impossible if a singularity develops at the corner point [1] and the wall behind the corner is rectilinear. A solution adjacent to the centered rarefaction wave downstream is found ineffective in the neighborhood of the singular characteristic emerging from the corner point. A uniformly effective solution is obtained using the method of deformed coordinates, and a shock wave is constructed.

1. Flow ahead the shock wave. Let us consider a steady plane subsonic flow past a perfect gas with the ratio of specific heats γ assuming that the walls of the finite convex corner are rectilinear (Fig. 1). A local solution for a finite value of the corner angle β in the region AOC is known [2, 3]. The problem therefore is that of constructing a solution in the region COD. Without affecting the general character of the arguments that follow, we can assume that the flow to the left of the last characteristic OC emerging from the corner point O, is potential. Then the solution [3] in the zone BOC of the centered rarefaction wave can be written in the form (z is fixed) [4]

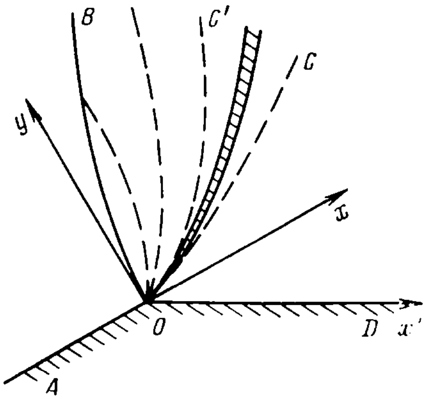

Fig. 1

$$\Phi = g_0(z)\, y + g_k(z)\, y^{1+2k/3}, \quad y \to 0, \quad z = x / y \tag{1.1}$$

$$g_0 = \nu^{-1} (1 + z^2)^{1/2} \sin \omega, \quad \omega = \nu \operatorname{arctg} z, \quad \nu^2 = (\gamma - 1) / (\gamma + 1)$$

$$g_m = (\sin \omega)^{m/3} (\cos \omega)^{1+m/3\nu^2} (1 + z^2)^{1/2+m/3} [A_m + H_m(\omega)]$$

$$H_m = \int (\sin \omega)^{-m/3} (\cos \omega)^{-1-m/3\nu^2} E_m(\omega)\, d\omega$$

$$E_m = -(\gamma + 1)(1 + 2m/3)^{-1} (\sin 2\omega)^{-1} (1 + z^2)^{1/2-m/3} G_m(\omega)$$

$$U = u_0(z) + u_k(z) y^{2k/3}, \quad V = v_0(z) + v_k(z)\, y^{2k/3}$$

$$u_m = g_m', \quad v_m = (1 + 2m/3)\, g_m - z g_m', \qquad m = 0, 1, \ldots$$

Here Φ, U and V are respectively the potential and the components of the velocity vector $\mathbf{w}$ along the x- and y-axes; $k = 1, 2, \ldots$ denotes summation; G_m are functions of the previous approximations $(G_1 \equiv 0)$; A_m are constants which are found by combining with a solution of the type given in [2]. In particular, we have

$$A_1 = -\frac{27}{5} C^{1/3} (\gamma + 1)^{1/18} (\gamma - 1)^{-1/6}, \quad A_2 = 0$$

where C is an arbitrary constant depending on the solution of the problem in the whole.

To simplify the boundary conditions and the solution in the region COD, we pass to the rectangular x'-, y'-coordinates obtained from the x-, y-coordinates by rotating them through an angle β (see Fig. 1). Then, with the angular coefficient z_c of the characteristic OC at the point O known and representing a root of the equation

$$z_c - g_0(z_c) / g_0'(z_c) = \operatorname{tg} \beta$$

we can rewrite the solution (1.1) in the form

$$\Phi = q_0^\circ(\xi)_x + q_k^\circ(\xi)\, x^{1+2k/3}, \quad x \to 0, \quad \xi = By / x \tag{1.2}$$

$$q_m^\circ = g_m \eta^{1+2m/3}, \quad \eta(\xi) = \xi_{B^{-1}} \cos\beta - \sin\beta, \quad m = 0, 1, \ldots$$

$$B = \nu^{-1} \operatorname{tg} \omega_c, \quad \omega_c = \nu \operatorname{arctg} z_c$$

(the primes are omitted). From (1.2) we obtain the equation of the characteristic OC in the form

$$\xi = 1 + \xi_k^\circ x^{2k/3} \tag{1.3}$$

Here

$$\frac{\xi_m^\circ}{U_0} = \frac{3(\gamma - 1)}{m} (\sin 2\omega_c)^{-2} \left[\frac{\sin 2\omega_c}{(\gamma^2 - 1)^{1/2}} v_m^\circ - u_m^\circ \right]_{\xi=1} + e_m^\circ(1) \tag{1.4}$$

$$u_m^\circ = (u_m \cos\beta - v_m \sin\beta)\, \eta^{2m/3}$$

$$v_m^\circ = (u_m \sin\beta + v_m \cos\beta)\, \eta^{2m/3}, \quad m = 1, 2, \ldots$$

where $U_0 = u_0^\circ(1)$ denotes the velocity of a homogeneous supersonic flow adjacent to the simple centered rarefaction wave, and $e_m^\circ(\xi)$ are functions of the preceding approximations $(e_1^\circ \equiv 0)$.

In the first approximation the velocity potential at the characteristic OC is given by

$$(\Phi)_{OC} = U_0 x + q_1^\circ(1)\, x^{5/3} + O(x^{7/3}) \tag{1.5}$$

We assume that the further extension of the flow is shock-free. Then the solution of the problem within COD must be solved with the data at the characteristic OC and on the wall OD (flow past condition). The solution with $x \to 0$ and fixed ξ must have the form (1.5) where $q_1^\circ(1)$ is replaced by $q_1(\xi)$ satisfying the equation

$$(1 - \xi^2) q_1'' + {}^4/_3\, \xi q_1' - {}^{10}/_9\, q_1 = 0$$

the general solution of which is

$$q_1 = \Lambda_1 (1 - \xi)^{5/3} + \Lambda_2 (1 + \xi)^{5/3} \tag{1.6}$$

where Λ_1 and Λ_2 are arbitrary constants.

From (1.6) it follows that $q_1''(\xi) \to \infty$ as $\xi \to 1$ if only $\Lambda_1 \neq 0$. For a rectilinear wall OD the coefficient Λ_1 cannot be zero, consequently infinite accelerations arise on the straight line $\xi = 1$ and a shock-free flow which formally exists becomes devoid of physical sense. This was shown in [5, 6] for the case of $\beta \ll 1$ by studying the transonic equations.

2. Flow behind the shock wave. We shall now assume that a curved shock wave, the form of which is to be determined, serves as a boundary separating the regions BOC and COD. We write the conditions at the discontinuity in the form

$$[w_\tau] = 0, \quad w_n{}^\circ w_n = 1 - \nu^2 w_\tau{}^2 \tag{2.1}$$

Here w_n and w_τ are the velocity vector $\mathbf{w}$ components normal and tangential to the discontinuity and $[X] = X - X^\circ$ denotes the jump in the value of X during the passage through the discontinuity. As the initial system of equations we use the transformed continuity and vorticity equations

$$\operatorname{div}\,[(1 - w^2)^{1/(\gamma-1)}\,\mathbf{w}] = 0 \tag{2.2}$$

$$\operatorname{rot}\left[\frac{\mathbf{w}\times\operatorname{rot}\mathbf{w}}{1 - w^2}\right] = 0$$

Analyzing the first boundary condition of (2.1) we find that the solution of (2.2) should be sought in the form

$$u = U_0 + u_k(\xi)\,x^{2k/3}, \quad v = v_k(\xi)\,x^{2k/3} \tag{2.3}$$

where u and v are components of the velocity vector along the axes of the new coordinate system. According to the flow past condition we have

$$v_m(0) = 0, \quad m = 1, 2, \ldots$$

From the first condition of (2.1) and from (2.3) it follows directly that the shock intensity at the corner apex is zero. The coefficients u_m and v_m $(m = 1, 2, \ldots)$ satisfy the following system of ordinary differential equations:

$$v_m' - B\,({}^2/_3\,mu_m - \xi u_m') = BF_m(\xi)$$

$${}^2/_3 m v_m - \xi v_m' - Bu_m' = BP_m(\xi)$$

where F_m and P_m are functions of the preceding approximations $(F_1 = P_1 = P_2 \equiv 0)$. Assuming that

$$u_m = \left(1 + \frac{2}{3}m\right) q_m(\xi) - \xi q_m'(\xi) - \int_0^\xi P_m(\xi)\,d\xi \tag{2.4}$$

$$v_m = Bq_m'(\xi)$$

we obtain the following equation for determining q_m:

$$(1 - \xi^2)\,q_m'' + \frac{4}{3}m\xi q_m' - \frac{2}{3}m\left(1 + \frac{2}{3}m\right)q_m = \tag{2.5}$$

$$F_m + \xi P_m - \frac{2}{3}m\int_0^\xi P_m(\xi)\,d\xi$$

The general solution of the homogeneous equation corresponding to (2.5) has the form

$$q_m(\xi) = \Lambda_{1m}(1 - \xi)^{1+2m/3} + \Lambda_{2m}(1 + \xi)^{1+2m/3}$$

Knowing this solution we can write the general solution of (2.5). After satisfying the flow past condition, we obtain

$$q_1 / U_0 = C_1 (\lambda^{5/3} + \mu^{5/3}), \quad \lambda = 1 - \xi, \quad \mu = 1 + \xi \tag{2.6}$$

$$q_2 / U_0 = C_2 (\lambda^{7/3} + \mu^{7/3}) + a_k \lambda^{(k+1)/3} \mu^{(6-k)/3}, \quad k = 1, 2, 3, 4 \tag{2.7}$$

$$a_1 = a_4 = -{}^5/_{12} [\gamma + 1 + (\gamma - 3) B^2] DC_1$$

$$a_2 = a_3 = -{}^{25}/_{72} (\gamma + 1) M^2 D C_1$$

$$D = M^2 C_1 / B^2, \quad M^2 = 1 + B^2$$

$$q_3 / U_0 = C_3 (\lambda^3 + \mu^3) - {}^9/_{10}\, c\, (\lambda^{-1/3} \mu^{10/3} + \lambda^{10/3} \mu^{-1/3}) + b_k \lambda^{k/3} \mu^{(9-k)/3}, \quad k = 1, 2, \ldots, 8; \quad P_3(\xi) = 2C_4 \xi \tag{2.8}$$

$$c = -{}^2/_9 a_1^2 / C_1, \quad b_1 = b_8 = {}^4/_5 a_1 a_2 / C_1$$

$$b_2 = b_7 = -\frac{D}{56} \left\{ -\frac{56 a_1 C_2}{DC_1} + \frac{10}{3} \left[17 (\gamma + 1) - (3\gamma + 11) B^2\right] a_1 + 10 (\gamma + 1) M^2 a_2 + \frac{125}{9} [\gamma + 1 + 2 (\gamma - 3) B^2 + (\gamma + 1) B^4] C_1^2 \right\}$$

$$b_3 = b_6 = -\frac{5D}{72} \left\{ 14 (\gamma + 1) M^2 C_2 + 2 [7 (\gamma + 1) - (\gamma - 3) B^2] a_2 + \frac{25}{9} (\gamma + 1) M^4 C_1^2 \right\}$$

$$b_4 = b_5 = -\frac{D}{40} \left\{ -\frac{56 a_1 C_2}{DC_1} + \frac{25}{3} (\gamma + 1)(5 + B^2) a_1 + \frac{40}{3} [2 (\gamma + 1) + \gamma B^2] a_2 + \frac{125}{9} (\gamma + 1)(1 - B^4) C_1^2 \right\}$$

The constants C_1, C_2, C_3 and C_4 appearing in (2.4),(2.6)−(2.8) are found from the conditions at the discontinuity.

The solution (2.3) does not hold in the region where $\lambda \sim x^2$. This is explained by the fact that $q_2'(\xi)$, $q_3(\xi) \to \infty$ for $\xi \to 1$ and the conditions at the discontinuity can no longer be met. The accumulation of singularities in the solution (2.3) can be prevented by deforming the coordinates [7, 8]. To do this we write the required solution in the parameteric form

$$u = U_0 + U_k(s)\, x^{2k/3}, \quad v = V_k(s)\, x^{2k/3}, \qquad \xi = s + \xi_k(s)\, x^{2k/3} \tag{2.9}$$

where the coefficients U_k, V_k and the deformation ξ_k are to be determined. The value of the parameter $s = 1$ corresponds to the special characteristic OC', which is the only characteristic in the region $C'OD$ which emerges from the corner point and moves to the left [9]. To find the solution in the form (2.9), we introduce an auxilliary function which is a velocity potential in an irrotational flow past the corner

$$\Phi = U_0 x + q_1(\xi)\, x^{5/3} + q_2(\xi)\, x^{7/3} + q_3(\xi)\, x^3 + O(x^{11/3}) \tag{2.10}$$

where the coefficients are given by the formulas (2.6)−(2.8). Using (2.10) we can write the solution (2.3) in the form

$$u = \Phi_x - B^2 C_4 y^2, \quad v = \Phi_y$$

and from this we conclude that the parametrization of (2.9) is equivalent to representing the function Φ in the form

$$\Phi = U_0 x + Q_1(s)\, x^{5/3} + Q_2(s)\, x^{7/3} + Q_3(s)\, x^3 + O(x^{11/3})$$

$$\xi = s + \xi_1(s)\, x^{2/3} + \xi_2(s)\, x^{4/3} + O(x^2)$$

Carrying out the re-expansion of the functions (2.10) as given by the method in [8] we find, that Q_1, Q_2 and Q_3 are determined by the formulas (2.6)—(2.8) in which ξ is replaced by s, and the right-hand sides of the first equations in (2.7), (2.8) complemented by the equations containing the deformations, namely

$$\xi_1 Q_1' / U_0, \quad (\xi_1 Q_2' + \xi_2 Q_1' - {}^1/_2 \xi^2{}_1 Q_1'') / U_0$$

Requiring that Q_2' and Q_3' be bounded when $s = 1$, we find

$$\xi_1 = {}^3/_5 2^{5/3} a_1 / C_1, \quad \xi_2 = {}^3/_5 2^{7/3} b_2 / C_1$$

All subsequent deformations can also be chosen as constants. In general we have

$$\frac{\xi_m}{U_0} = \frac{6(\gamma-1)}{2m+3}(\sin 2\omega_c)^{-2}\left[\frac{\sin 2\omega_c}{(\gamma^2-1)^{1/2}} V_m - U_m\right]_{s=1} + e_m(1) \qquad (2.11)$$

where $e_m(s)$ are functions of the preceding approximations ($e_1 \equiv 0$). The coefficients U_m and V_m of the expansions (2.9) have the following form for $m = 1, 2, 3$:

$$U_1 = {}^5/_3 Q_1 - sQ_1', \quad U_2 = {}^7/_3 Q_2 - sQ_2' - {}^5/_3 \xi_1 Q_1'$$
$$U_3 = 3Q_3 - sQ_3' - {}^5/_3 \xi_1 Q_2' - {}^7/_3 \xi_2 Q_1' - C_4 s^2, \quad V_m = BQ_m'$$

We can check by direct substitution that the flow past conditions of the wall OD hold

$$V_1(0) = 0, \quad V_2(0) = \xi_1 V_1'(0)$$
$$V_3(0) = \xi_1 V_2'(0) + \xi_2 V_1'(0) - {}^1/_2\, \xi^2{}_1 V_1''(0)$$

We assume that the equation of discontinuity has the form

$$\xi = 1 + \xi^\circ{}_1 x^{2/3} + \xi_2^\circ x^{4/3} + \delta_3 x^2 + O(x^{8/3})$$
$$(\lambda = -s_3 x^2 + O(x^{8/3}))$$

The coefficients ξ_1°, ξ_2° are found using (1.4). The unknowns δ_3 and s_3 are connected by the following relation:

$$\delta_3 - s_3 = \xi_3 \qquad (2.12)$$

The difference between the discontinuity and the characteristic is described by the coefficient s_3; ξ_3 denotes a third order deformation although it should not be obtained by studying the fourth order approximation since ξ_3 represents, on the other hand, a coefficient of the special characterisitc computed from the third approximation. Taking into account the conditions (2.1), we find from (1.4) and (2.11), as was to be expected, that $\xi_1 = \xi_1^\circ$, $\xi_2 = \xi_2^\circ$.

The second condition of (2.1) is satisfied identically in the first and second approximation, while in the third approximation it yields

$$\delta_3 = \xi_3^\circ - {}^3/_2 s_3 - {}^5/_{12} A s_3^{2/3} - {}^{25}/_{108} A^2 s_3^{1/3} \qquad (2.13)$$
$$A = \frac{(\gamma+1)M^4}{B^2} C_1$$

Setting $s_3 = K^3 A^3 / 27$, we reduce the system (2.12), (2.13) to a single equation

$$K^3 + \frac{1}{2}K^2 + \frac{5}{6}K + \frac{54}{5}\frac{\xi_3 - \xi_3^\circ}{A^3} = 0$$

which has a single positive real root. For $\beta \ll 1$ the above equation becomes

$$K^3 + {}^1/_2 K^2 + {}^5/_6 K - {}^{25}/_{36} = 0$$

with the approximate value of the root $K = 0.5132$.

The constants C_1 and C_4 are found in the form

$$C_1 = U_0^{2/3} [2g_0 (z_c)]^{-4/3} g_1 (z_c), \quad C_4 = 0$$

Since $C_1 < 0$, we also have $s_3 < 0$. Then from (2.13) it follows that the discontinuity lies on the left of the characterisitc OC. This was shown from the case $\beta \ll 1$ in [6] using the transonic equations. It can be directly established that the velocity behind the shock is supersonic, and we have

$$\frac{[w_n]}{w_n^\circ} = -\frac{8B^2}{(\gamma + 1) M^2} (\delta_3 - \xi_3^\circ) x^2 + O(x^{8/3})$$

Since the coefficient in front of x^2 is negative, it follows that the discontinuity constructed is a shock wave.

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